FUZZY LOGIC EXTENSION 1.0 IN NETLOGO 5.2

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    fuzzy:discrete-numeric-set-with-label *label* $[[x_1\ y_1]\ [x_2\ y_2]\ ...\ [x_n\ y_n]]$

fuzzy:piecewise-linear-set $[[x_1\ y_1]\ [x_2\ y_2]\ ...\ [x_n\ y_n]]$
    fuzzy:piecewise-linear-set-with-label *label* $[[x_1\ y_1]\ [x_2\ y_2]\ ...\ [x_n\ y_n]]$

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fuzzy:interval-with-points-set [[[a b] v] $[[x_1\ y_1]\ [x_2\ y_2]\ ...\ [x_n\ y_n]]$]]
    fuzzy:interval-with-points-set-with-label *lbl* [[[a b] v] $[[x_1\ y_1]...\ [x_n\ y_n]]$]]

RULES

fuzzy:truncate-rule (= fuzzy:rule) *list-of-2-items consequent-fuzzy:set*
fuzzy:prod-rule *list-of-2-items consequent-fuzzy:set*
fuzzy:min-truncate-rule (= fuzzy:and-rule) *list-of-2-item-lists consequent-fuzzy:set*
fuzzy:max-truncate-rule (= fuzzy:or-rule) *list-of-2-item-lists consequent-fuzzy:set*
fuzzy:min-prod-rule *list-of-2-item-lists consequent-fuzzy:set*
fuzzy:max-prod-rule *list-of-2-item-lists consequent-fuzzy:set*

Basic introduction to fuzzy sets

Fuzzy sets are generalizations of conventional sets (also called "crisp" sets).

Note that a conventional (crisp) set *S* can be defined by a Membership Function $MF_S(x)$ that specifies, for each possible element *x* in the universe *X* (or space of all objects), whether the element belongs to *S* or not. If an element *x* belongs to *S*, then the membership function of set *S* applied to element *x* equals 1. If an element *x* does not belong to *S*, then the membership function of set *S* applied to element *x* equals 0.

$$MF_S(x) = \begin{cases} 1 & if\ x \in S \\ 0 & if\ x \notin S \end{cases}$$

For instance, the crisp set *EvenNumbers* contains the elements {2, 4, 6, …}, so its membership function equals 1 when applied to each of such numbers (i.e. $MF_{EvenNumbers}(2) = 1$). By contrast, the membership function of the crisp set *EvenNumbers* equals 0 when applied to numbers such as 1, 3, or 5 (i.e. $MF_{EvenNumbers}(5) = 0$).

The sharp dichotomy in the concept of "belonging" in conventional sets (i.e. an element either belongs to the set, or it does not) is relaxed with fuzzy sets, which allow for partial membership. This can be useful to define and reason with fuzzy concepts, such as e.g. "Tall". We may be reluctant to specify a precise and definite threshold value *h* such that every adult person with a height below *h* is "not tall" and every adult person with a height greater or equal to *h* is "tall". This would imply, in particular, that a person with a height just below *h* would be definitely "not tall", but someone only 1 mm higher would be definitely "tall".

Even though our concept of "Tall" does not seem to be perfectly precise, we may still find the concept useful in logical reasoning, e.g. from the general premise

"*Parents that are tall tend to have children who will be tall*"

and the specific premise

"*Mary and John are tall*",

we may conclude that

"*Mary and John's child will likely be tall*".

Note that we, humans, have developed a remarkable (and very useful) tolerance for imprecision. We seem to feel fairly comfortable dealing with rules, instructions and logical statements defined with fuzzy concepts (e.g. "If you drive very fast, your fuel consumption will be high"); however, computers using Boolean logic may not find it so straightforward. Using fuzzy logic, we can use computers to model and exploit this type of fuzzy reasoning.

Fuzzy sets are implemented by allowing membership functions to give intermediate values in between 0 and 1. Thus, the membership function MF of a fuzzy set assigns to each element *x* of a certain universe *X* a degree of membership $MF(x) \in [0,1]$. This mapping is usually denoted:

$$MF(x): X \to [0,1]$$

As an example, one could define the fuzzy set *Tall*, with a membership function whose argument is a number $x \in [0,250]$ representing height measured in cm (see e.g. Fig. 1). The universe *X* (or space of all objects) in this example would be the continuous interval [0,250].[1]

---

[1] Remember that the *illusion* of continuity is implemented in most computer platforms (NetLogo, in particular) using floating point numbers (see http://ccl.northwestern.edu/netlogo/docs/programming.html#math).

Fig. 1. Membership function of a fuzzy set created running the code:
fuzzy:plot fuzzy:piecewise-linear-set [[0 0] [130 0.1] [180 0.9] [200 1] [250 1]]

The fuzzy set *Tall* shown in Fig. 1 would assign a degree of membership 0 to height 0 cm, a degree of membership 0.1 to height 130 cm, and a degree of membership 0.82 to height 175 cm.

The extension described in this document provides several procedures that assist in modeling fuzzy systems within NetLogo. In particular, using this fuzzy logic extension it will be straightforward to implement agent-based models where individual agents implement fuzzy rules such as: "If both parents are tall, then the son is tall".

How to include the fuzzy logic extension

To be able to use all the procedures explained in this document you must save the folder named fuzzy containing the file "Fuzzy.jar" in the same directory where your NetLogo code lies or in the extension's folder where NetLogo was installed[2]. Also, you will have to add the extensions keyword at the beginning of the code tab (before declaring any breeds or variables), followed by the word fuzzy in between square brackets, as follows:

```
extensions [fuzzy]
```

---

[2] See http://ccl.northwestern.edu/netlogo/docs/extensions.html for details.

Visualization

```
fuzzy:plot input-set
```

Procedure run by any agent (i.e. the observer, a turtle, a patch or a link); it draws the membership function of fuzzy set *input-set* on the current plot.

```
let my-set fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
set-current-plot "Fuzzy set"   ;; make sure this plot exists!
set-plot-x-range 0 6
fuzzy:plot my-set               ;; The membership function is plotted
                                ;; in the current plot. See Fig. 2 below.
```
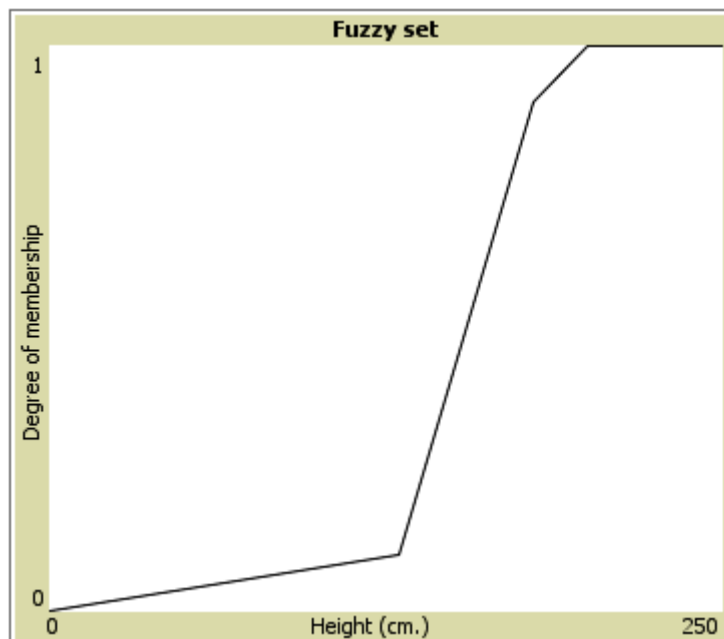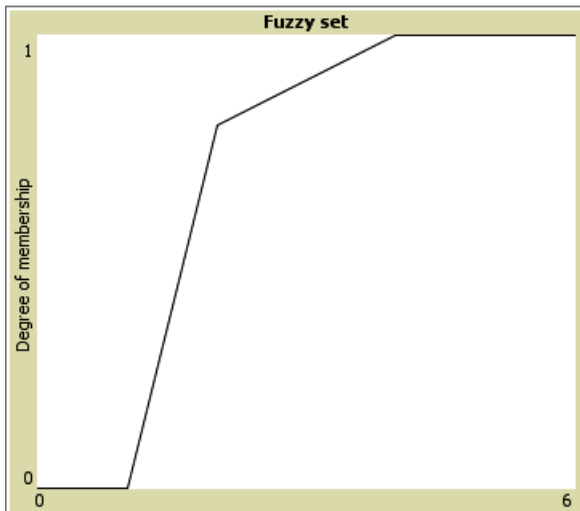

Fig. 2. Membership function of a piecewise linear fuzzy set created running the code:
fuzzy:plot fuzzy:piecewise-linear-set
[[0 0][1 0][2 0.8][4 1][6 1]]
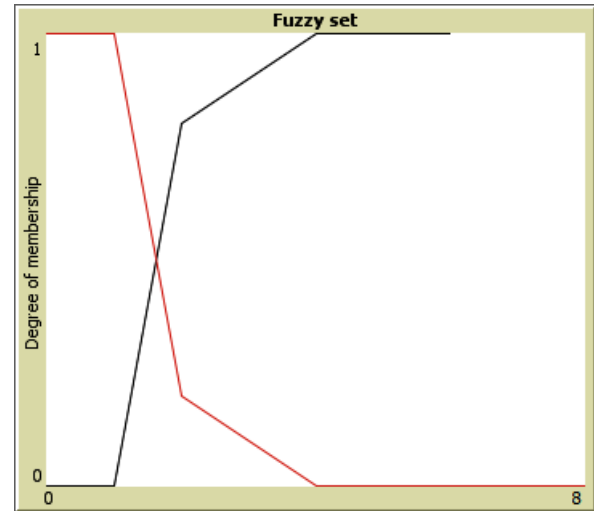
Fig. 3. Membership functions of two piecewise linear fuzzy sets. By default, the procedure "fuzzy:plot" does not clear the plot. The color of the plot can be changed using the primitive "set-plot-pen-color".

By default, the procedure **fuzzy:plot** will not clear the graph. This is done to allow you to plot several membership functions on the same graph.

```
let my-set1 fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
let my-set2 fuzzy:piecewise-linear-set [[0 1][1 1][2 0.2][4 0][8 0]]
set-plot-x-range 0 8
fuzzy:plot my-set1              ;; The membership function is the current plot
set-plot-pen-color red         ;; changes the color of the pen we use to plot
fuzzy:plot my-set2             ;; See Fig. 3.
clear-plot                     ;; clears the plot
```

Evaluation of membership functions

Fuzzy sets can be evaluated using a procedure named **fuzzy:evaluation-of**, which computes the degree of membership of any input value (either a number or another fuzzy set) on the set.

```
fuzzy:evaluation-of input-set x
```

Reporter run by any agent; it returns the value of the Membership Function of the set *input-set* on input value *x* (either a number or another fuzzy set). The procedure **fuzzy:evaluation-of** can also take a list as second argument; in that case, it returns a list with the evaluation of the Membership Function of the set *input-set* on each item in the input list (provided as second argument).

Let us start assuming *x* is a number. If the membership function is not defined at input number *x*, the procedure **fuzzy:evaluation-of** returns the value "NaN" (Not a Number).

```
let my-set fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
fuzzy:plot my-set                          ;; This produces Fig. 2.
show fuzzy:evaluation-of my-set 1.5        ;; this shows 0.4
show fuzzy:evaluation-of my-set [1.5 2 3 4 7]   ;; shows [0.4 0.8 0.9 1 NaN]
let a fuzzy:evaluation-of my-set [1.5 2 3 4 7]  ;; a equals [0.4 0.8 0.9 1 NaN]
```

The input value *x* can also be another fuzzy set. In that case, **fuzzy:evaluation-of** returns the degree of fulfilment between the two sets, i.e. the maximum value *y* such that there exists a number with degree of membership greater or equal to *y* in both sets. This also works if the input is a list of fuzzy sets.

```
let my-set1 fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
let my-set2 fuzzy:piecewise-linear-set [[0 1][1 1][2 0.2][4 0][8 0]]
fuzzy:plot my-set1     ;; draws the membership function in the current plot
set-plot-pen-color red         ;; changes the color of the pen we use to plot
fuzzy:plot my-set2             ;; See Fig. 3.
show fuzzy:evaluation-of my-set1 my-set2              ;; this shows 0.5
show fuzzy:evaluation-of my-set2 my-set1              ;; this shows 0.5
show fuzzy:evaluation-of my-set1 list my-set1 my-set2   ;; this shows [1 0.5]
```

If there is no number at which the two sets are defined, the procedure **fuzzy:evaluation-of** returns the value "NaN" (Not a Number)

```
let my-set1 fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
let my-set2 fuzzy:piecewise-linear-set [[7 1][8 0.5]]
show fuzzy:evaluation-of my-set1 my-set2                 ;; this shows NaN
show fuzzy:evaluation-of my-set2 my-set1                 ;; this shows NaN
show fuzzy:evaluation-of my-set1 (list my-set1 my-set2)   ;; this shows [1 NaN]
```

Finally, the input list can contain numbers and sets.

```
let my-set1 fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
let my-set2 fuzzy:piecewise-linear-set [[7 1][8 0.5]]
show fuzzy:evaluation-of my-set1 (list my-set1 my-set2 0 2 3 10)
                                   ;; this shows [1 NaN 0 0.8 0.9 NaN]
```

Retrieval of fuzzy sets

1. A variable (or several variables) can store a pointer to a fuzzy set.

```
let one-var fuzzy:piecewise-linear-set [[0 0][1 0][2 0.8][4 1][6 1]]
let another-var one-var              ;; another-var points to the new fuzzy set
fuzzy:plot another-var               ;; This produces Fig. 2.
show one-var = another-var           ;; this shows true
```

2. You can also retrieve a fuzzy set if you know its label, by using the procedure **fuzzy:set-with-label**. Labels are explained in detail in the next section.

```
fuzzy:set-with-label label
```

Reports the fuzzy set that has the label equal to the parameter `label`.

```
let one-var fuzzy:piecewise-linear-set-with-label "hot" [[0 0][40 1]]
show fuzzy:set-with-label "hot"
                    ;; {{fuzzylogic:PiecewiseLinear [[0 0][40 1]]}} is shown
fuzzy:plot fuzzy:set-with-label "hot"     ;; The membership function is
                                          ;; drawn in the current plot.
show one-var = fuzzy:set-with-label "hot"  ;; this shows true
```

Labels

Labels are strings that can be assigned to fuzzy sets to facilitate their retrieval (as shown in the previous section). Any particular label can be assigned to at most one fuzzy set, and any particular fuzzy set can have at most one label. The label can be assigned at the moment the fuzzy set is created or at any later stage using the procedure **fuzzy:set-label-of**.

```
fuzzy:set-label-of input-set string
```

If a fuzzy set is assigned a label that had been previously assigned to another fuzzy set, then a message notifying this anomaly is issued, and the label is exclusively assigned to the latter fuzzy set.

```
let my-set fuzzy:gaussian-set-with-label "low" [0 50 [0 150]]
show my-set = fuzzy:set-with-label "low"           ;; this shows true
let another-set fuzzy:piecewise-linear-set [[7 1][8 0]]
fuzzy:set-label-of another-set "low"               ;; a message is issued
show my-set = fuzzy:set-with-label "low"           ;; this shows false
show another-set = fuzzy:set-with-label "low"      ;; this shows true
show fuzzy:set-with-label "low"
                    ;; {{fuzzylogic:PiecewiseLinear [[7 1][8 0]]}} is shown
```

To clear all labels in a model, you can use:

```
fuzzy:clear-all-labels
```

This method is useful because fuzzy sets that have a label assigned to them stay in memory even after executing clear-all. Thus, it can be useful to call **fuzzy:clear-all-labels** at the beginning of your program.

To clear the label of a fuzzy set, you can use:

```
fuzzy:clear-label-of input-set
```

```
fuzzy:clear-all-labels
let my-set fuzzy:gaussian-set-with-label "small" [0 50 [0 150]]
let another-set fuzzy:piecewise-linear-set [[7 1][8 0]]
fuzzy:clear-label-of my-set                 ;; label "small" is cleared
fuzzy:set-label-of another-set "small"      ;; no message is issued
show fuzzy:set-with-label "small"
                        ;; {{fuzzylogic:PiecewiseLinear [[7 1][8 0]]}} is shown
```

To clear a certain label, you can use:

```
fuzzy:clear-label string
```

```
fuzzy:clear-all-labels
let my-set fuzzy:gaussian-set-with-label "small" [0 50 [0 150]]
let another-set fuzzy:piecewise-linear-set [[7 1][8 0.5]]
fuzzy:clear-label "small"                   ;; label "small" is cleared
fuzzy:set-label-of another-set "small"      ;; no message is issued
```

To get the label of a fuzzy set, you can use:

```
fuzzy:label-of input-set
```

To check whether a fuzzy-set has a label or not, you can use:

```
fuzzy:has-label? input-set
```

Finally, to check whether a certain label exists or not, you can use:

```
fuzzy:label-exists? string
```

```
fuzzy:clear-all-labels
let my-set fuzzy:gaussian-set-with-label "low" [0 50 [0 150]]
show fuzzy:has-label? my-set          ;; true is shown
show fuzzy:label-of my-set            ;; "low" is shown
show fuzzy:label-exists? "low"        ;; true is shown
```

Creation of fuzzy sets from scratch

<div style="border:1px solid #999;padding:4px">

`fuzzy:discrete-numeric-set` [[x₁ y₁] [x₂ y₂] ... [xₙ yₙ]]

</div>

Creates and reports a new discrete fuzzy set. The parameter must be a list of numbers of the form: $[[x_1,y_1] [x_2,y_2] ... [x_n,y_n]]$, i.e., a list of 2-number lists. The membership function of the reported fuzzy set is a discrete function that assigns the degree of membership y_i to input x_i. For values of $x \neq x_i$, the membership function is not defined.

```
let my-set fuzzy:discrete-numeric-set [[1 0.3][2 0.8][4 0]]
show fuzzy:evaluation-of my-set 2            ;; my-set shows 0.8
show fuzzy:evaluation-of my-set 3            ;; NaN is shown
fuzzy:plot my-set                            ;; This produces Fig. 4 below.
```
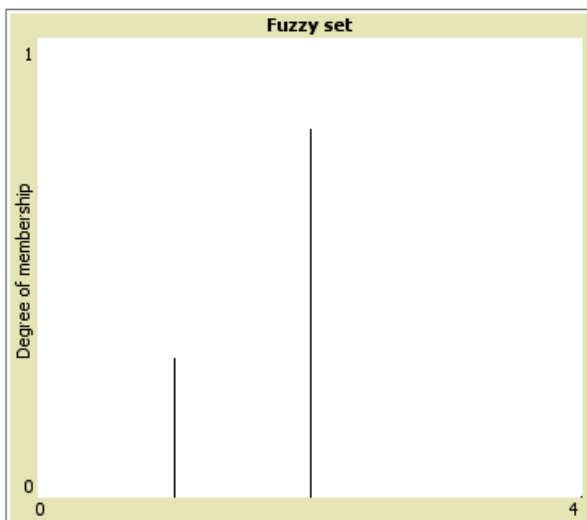

Fig. 4. Membership function of a discrete fuzzy set created running the code:
fuzzy:plot fuzzy:discrete-numeric-set [[1 0.3][2 0.8][4 0]]
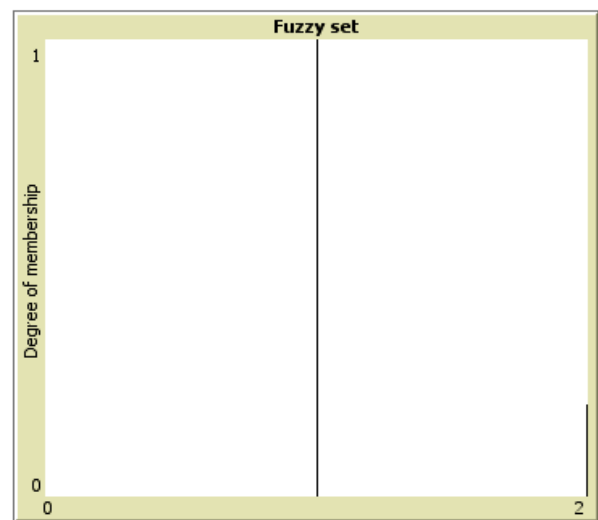
Fig. 5. Membership function of a discrete fuzzy set created running the code:
fuzzy:plot fuzzy:discrete-numeric-set [[1 1][2 0.2]]

<div style="border:1px solid #999;padding:4px">

`fuzzy:discrete-numeric-set-with-label` *label* [[x₁ y₁] [x₂ y₂] ... [xₙ yₙ]]

</div>

Creates and reports a new discrete fuzzy set with label *label*. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:discrete-numeric-set**.

```
let my-set fuzzy:discrete-numeric-set-with-label "low" [[1 1][2 0.2]]
show fuzzy:evaluation-of fuzzy:set-with-label "low" 1        ;; my-set shows 1
show fuzzy:evaluation-of fuzzy:set-with-label "low" 3
                                             ;; NaN is shown
show my-set = fuzzy:set-with-label "low"     ;; true is shown
fuzzy:plot my-set                            ;; This produces Fig. 5 above.
```

```
fuzzy:piecewise-linear-set [[x₁ y₁] [x₂ y₂] ... [xₙ yₙ]]
```

Creates and reports a new piecewise linear fuzzy set. The parameter must be a list of numbers of the form: $[[x_1,y_1]\ [x_2,y_2]\ ...\ [x_n,y_n]]$, i.e., a list of 2-number lists. The membership function of the reported fuzzy set is a piecewise linear function that joins all the points $[x_i,y_i]$ (previously sorted on x_i).

```
let my-set fuzzy:piecewise-linear-set [[0 0][1 0][1 0.3][2 0.8][4 0]]
fuzzy:plot my-set                             ;; This produces Fig. 6 below.
show fuzzy:evaluation-of my-set 2             ;; 0.8 is shown
show fuzzy:evaluation-of my-set 3             ;; 0.4 is shown
show fuzzy:evaluation-of my-set 5             ;; NaN is shown
```
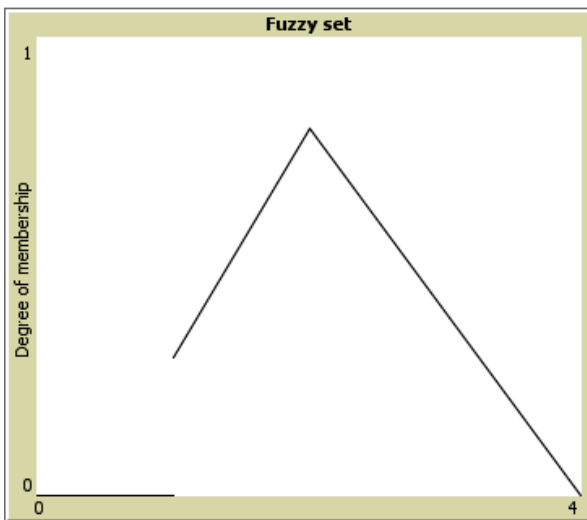

Fig. 6. Membership function of a piecewise linear fuzzy set created running the code: fuzzy:plot fuzzy:piecewise-linear-set [[0 0][1 0][1 0.3][2 0.8][4 0]]
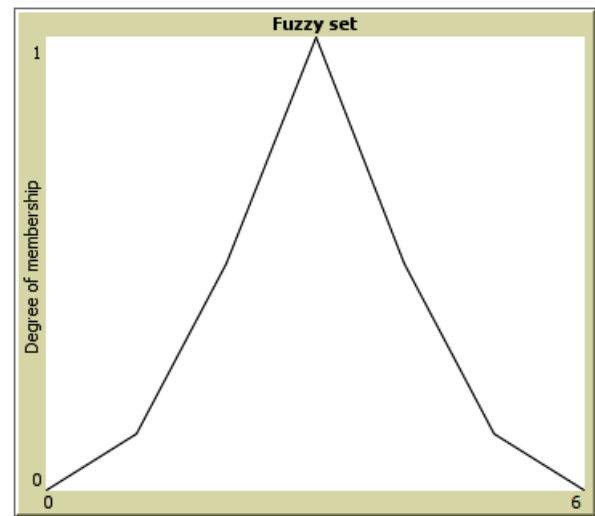
Fig. 7. Membership function of a piecewise linear fuzzy set created running the code: fuzzy:plot fuzzy:piecewise-linear-set [[0 0][1 0.125] [2 0.5][3 1][4 0.5][5 0.125][6 0]]

```
fuzzy:piecewise-linear-set-with-label label [[x₁ y₁] [x₂ y₂] ... [xₙ yₙ]]
```

Creates and reports a new piecewise linear fuzzy set with label *label*. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:piecewise-linear-set**.

```
let my-set fuzzy:piecewise-linear-set-with-label "about 3"
                    [[0 0][1 0.125][2 0.5][3 1][4 0.5][5 0.125][6 0]]
set-plot-x-range 0 6
fuzzy:plot my-set                             ;; This produces Fig. 7 above.
show fuzzy:evaluation-of fuzzy:set-with-label "about 3" 2.5
                                              ;; my-set shows 0.75
show fuzzy:evaluation-of fuzzy:set-with-label "about 3" 0    ;; 0 is shown
show my-set = fuzzy:set-with-label "about 3"           ;; true is shown
```

```
fuzzy:trapezoidal-set [a b c d e f height]
```

Creates and reports a new piecewise linear fuzzy set with the shape of trapezoid. The parameter must be a list of numbers of the form: [*a b c d e f height*]. The membership function of the reported fuzzy set is a piecewise linear function that equals 0 in the range *a* to *b*, increases linearly from 0 to *height* in the range *b* to *c*, is equal to *height* in the range *c* to *d*, decreases linearly from *height* to 0 in the range *d* to *e*, and equals 0 in the range *e* to *f*. The universe is [*a*, *f*]. For values outside the universe, the membership function is not defined. Note that the membership function at the extreme values *a* and *f* is 0. Thus, do not use this procedure to create membership functions that are meant to equal a value other than 0 at the extremes. In that case, use **fuzzy:piecewise-linear-set** instead.

```
let my-set fuzzy:trapezoidal-set [1 2 6 7 9 10 0.7]
set-plot-x-range 1 10
fuzzy:plot my-set                          ;; This produces Fig. 8 below.
show fuzzy:evaluation-of my-set 3          ;; 0.175 is shown
show fuzzy:evaluation-of my-set 6.5        ;; 0.7 is shown
show fuzzy:evaluation-of my-set 11         ;; NaN is shown
```


Fig. 8. Membership function of a trapezoidal fuzzy set created running the code:
fuzzy:plot fuzzy:trapezoidal-set [1 2 6 7 9 10 0.7]
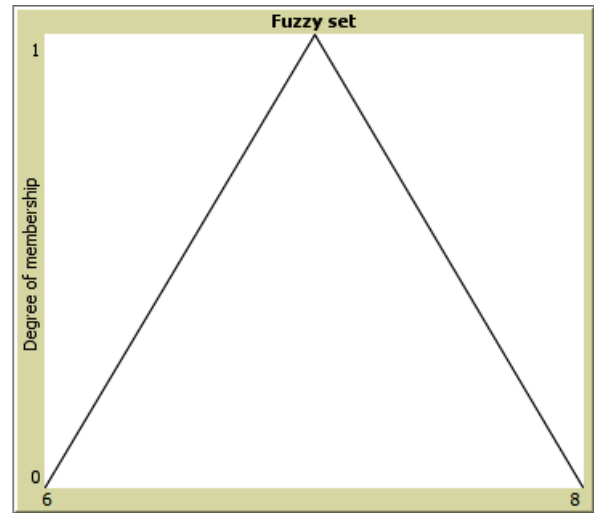
Fig. 9. Membership function of a trapezoidal fuzzy set created running the code:
fuzzy:plot fuzzy:trapezoidal-set [6 6 7 7 8 8 1]

```
fuzzy:trapezoidal-set-with-label label [a b c d e f height]
```

Creates and reports a new trapezoidal fuzzy set with label `label`. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:trapezoidal-set**.

```
let my-set fuzzy:trapezoidal-set-with-label "close to 7" [6 6 7 7 8 8 1]
set-plot-x-range 6 8
fuzzy:plot fuzzy:set-with-label "close to 7"        ;; This produces Fig. 9
show fuzzy:evaluation-of fuzzy:set-with-label "close to 7" 6.25
                                                    ;; my-set shows 0.25
show fuzzy:evaluation-of fuzzy:set-with-label "close to 7" 7.25
                                                    ;; 0.75 is shown
show my-set = fuzzy:set-with-label "close to 7"     ;; true is shown
```

```
fuzzy:logistic-set [x₀ a b [lower-limit upper-limit]]
```

Creates and reports a new logistic fuzzy set. The parameter must be a list of the form: [x_0 *a b* [*lower-limit upper-limit*]]. The membership function of the reported fuzzy set is the logistic function:

$$MF(x) = \frac{1}{1 + a \cdot e^{-b(x-x_0)}}$$

within the range [*lower-limit upper-limit*] for x. For values of *x* outside the range (universe), the membership function is not defined.

```
let my-set fuzzy:logistic-set [2 1 1 [-6 10]]
fuzzy:plot my-set                              ;; This produces Fig. 10 below.
show fuzzy:evaluation-of my-set 0              ;; 0.11920292202211755 is shown
show fuzzy:evaluation-of my-set 2              ;; 0.5 is shown
show fuzzy:evaluation-of my-set 4              ;; 0.8807970779778823 is shown
show fuzzy:evaluation-of my-set 8              ;; 0.9975273768433653 is shown
show fuzzy:evaluation-of my-set 11             ;; NaN is shown
```
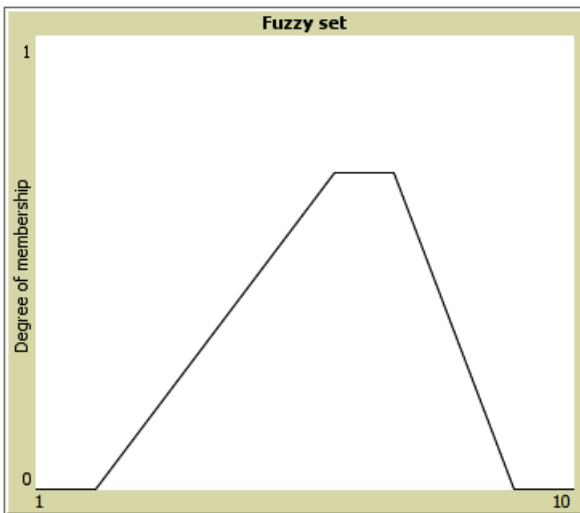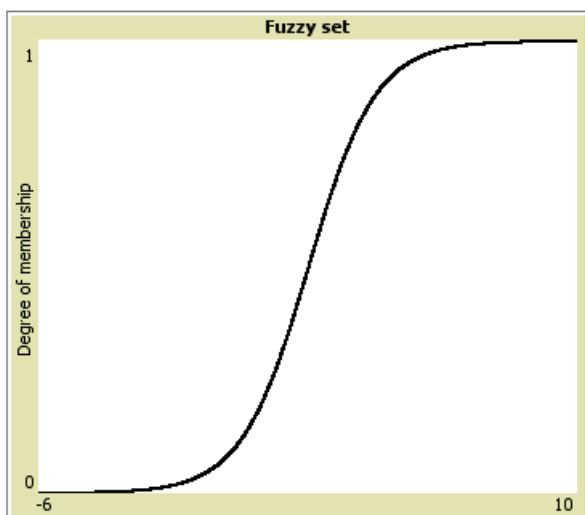

Fig. 10. Membership function of a logistic fuzzy set created running the code: fuzzy:plot fuzzy:logistic-set [2 1 1 [-6 10]]
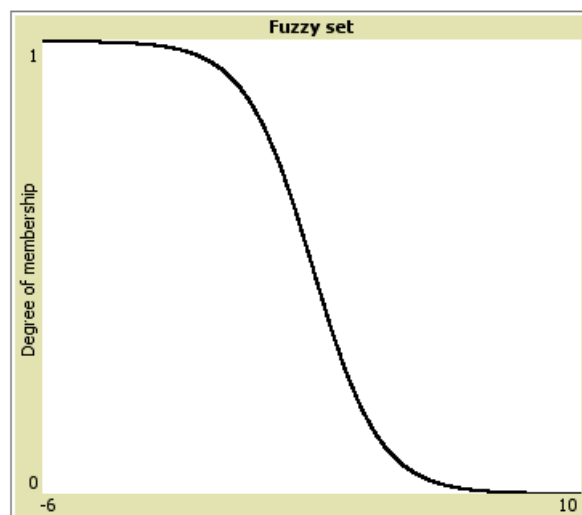
Fig. 11. Membership function of a logistic fuzzy set created running the code: fuzzy:plot fuzzy:logistic-set [2 1 -1 [-6 10]]

```
fuzzy:logistic-set-with-label label [x₀ a b [lower-limit upper-limit]]
```

Creates and reports a new logistic fuzzy set with label `label`. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:logistic-set**.

```
let my-set fuzzy:logistic-set-with-label "low" [2 1 -1 [-6 10]]
fuzzy:plot fuzzy:set-with-label "low"          ;; This produces Fig. 11 above.
show fuzzy:evaluation fuzzy:set-with-label "low" 4    ;; my-set shows 0.119202
show fuzzy:evaluation fuzzy:set-with-label "low" 2    ;; 0.5 is shown
show fuzzy:evaluation fuzzy:set-with-label "low" 0    ;; 0.880797 is shown
show fuzzy:evaluation fuzzy:set-with-label "low" -4   ;; 0.997527 is shown
show fuzzy:evaluation fuzzy:set-with-label "low" -7   ;; NaN is shown
show my-set = fuzzy:set-with-label "low"             ;; true is shown
```

```
fuzzy:gaussian-set [m s [lower-limit upper-limit]]
```

Creates and reports a new Gaussian (bell-shaped) fuzzy set. The parameter must be a list of the form: [*m s* [*lower-limit upper-limit*]], where *m* plays a role similar to the mean and *s* to the standard deviation.[3] Specifically, the membership function of the reported fuzzy set is the Gaussian function:

$$MF(x) = e^{-\frac{(x-m)^2}{2s^2}}$$

within the range [*lower-limit upper-limit*] for x. Note that the value of the function at *x = m* is always 1, so it is a scaled normal distribution. For values of *x* outside the range (universe), the membership function is not defined.

```
let my-set fuzzy:gaussian-set [5 2 [0 10]]
fuzzy:plot my-set                         ;; This produces Fig. 12 below.
show fuzzy:evaluation-of my-set 0         ;; 0.04393693362340742 is shown
show fuzzy:evaluation-of my-set 2         ;; 0.32465246735834974 is shown
show fuzzy:evaluation-of my-set 5         ;; 1 is shown
```


Fig. 12. Membership function of a gaussian fuzzy set created running the code:
fuzzy:plot fuzzy:gaussian-set [5 2 [0 10]]

```
fuzzy:gaussian-set-with-label label [x₀ a b [lower-limit upper-limit]]
```

Creates and reports a new gaussian fuzzy set with label `label`. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:gaussian-set**.

```
let my-set fuzzy:gaussian-set-with-label "normal" [5 2 [0 10]]
fuzzy:plot fuzzy:set-with-label "normal"          ;; This produces Fig. 12
show fuzzy:evaluation-of fuzzy:set-with-label "normal" 3
                                                  ;; my-set shows 0.60653
show fuzzy:evaluation-of fuzzy:set-with-label "normal" 7      ;; 0.60653 is shown
show my-set = fuzzy:set-with-label "normal"                   ;; true is shown
```

---

[3] Note that the membership function is only defined in the range [*lower-limit upper-limit*] and *MF(m)* = 1.

```
fuzzy:exponential-set [a b c [lower-limit upper-limit]]
```

Creates and reports a new exponential fuzzy set. The parameter must be a list of the form: [*a b c* [*lower-limit upper-limit*]]. The membership function of the reported fuzzy set is the exponential function:

$$MF(x) = a \cdot e^{b(x-c)}$$

within the range [*lower-limit upper-limit*] for *x*; *MF(x)* is clipped so its value is always contained in the interval [0,1]. Note that the value of the function at *x = c* is equal to *a* (assuming $a \in [0,1]$). For values of *x* outside the range (universe), the membership function is not defined.

```
let my-set fuzzy:exponential-set [0.5 1 5 [0 10]]
fuzzy:plot my-set                              ;; This produces Fig. 13 below.
show fuzzy:evaluation-of my-set 0              ;; 0.0033689734995427335 is shown
show fuzzy:evaluation-of my-set 4              ;; 0.18393972058572117 is shown
show fuzzy:evaluation-of my-set 5              ;; 0.5 is shown
show fuzzy:evaluation-of my-set 6              ;; 1 is shown
show fuzzy:evaluation-of my-set 11             ;; NaN is shown
```
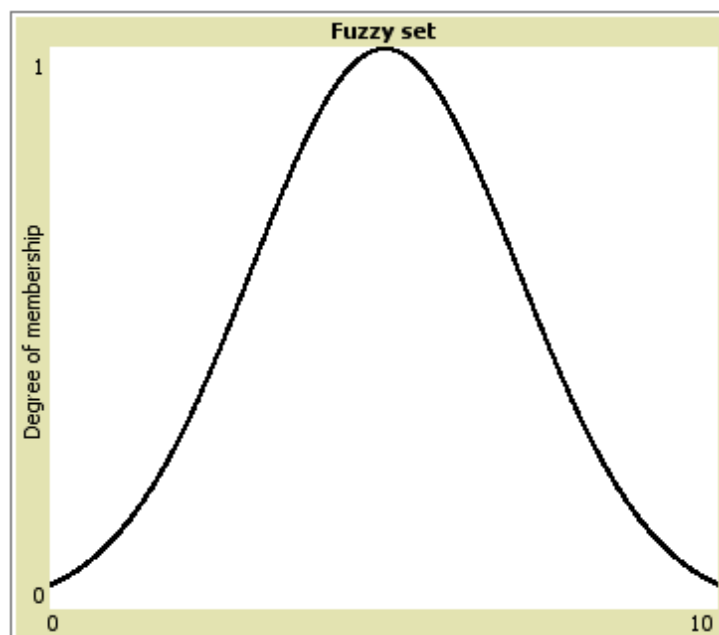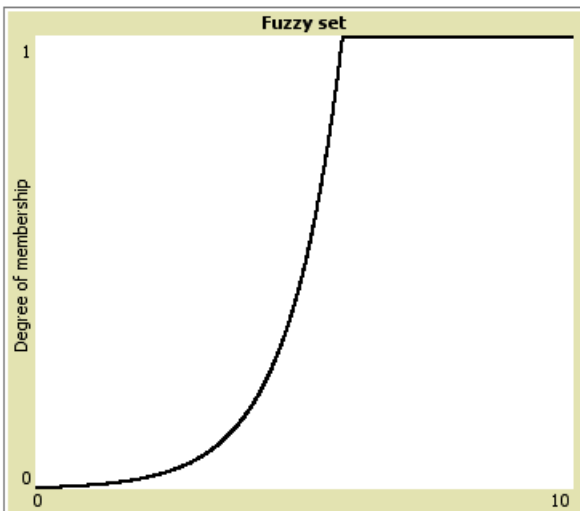

Fig. 13. Membership function of a logistic fuzzy set created running the code:
fuzzy:plot fuzzy:exponential-set [0.5 1 5 [0 10]]
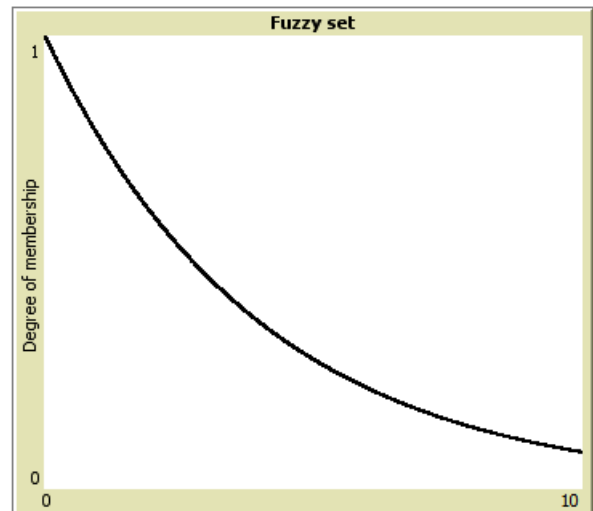
Fig. 14. Membership function of a logistic fuzzy set created running the code:
fuzzy:plot fuzzy:exponential-set [1 -0.25 0 [0 10]]

```
fuzzy:exponential-set-with-label label [a b c [lower-limit upper-limit]]
```

Creates and reports a new exponential fuzzy set with label `label`. The parameter `label` can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:exponential-set**.

```
let my-set fuzzy:exponential-set-with-label "low" [1 -0.25 0 [0 10]]
fuzzy:plot fuzzy:set-with-label "low"          ;; This produces Fig. 14 above
show fuzzy:evaluation-of fuzzy:set-with-label "low" 0      ;; my-set shows 1
show fuzzy:evaluation-of fuzzy:set-with-label "low" 4         ;; 0.368 is shown
show fuzzy:evaluation-of fuzzy:set-with-label "low" 5         ;; 0.287 is shown
show fuzzy:evaluation-of fuzzy:set-with-label "low" 6         ;; 0.223 is shown
show fuzzy:evaluation-of fuzzy:set-with-label "low" 11        ;; NaN is shown
show my-set = fuzzy:set-with-label "low"                      ;; true is shown
```

```
fuzzy:interval-with-points-set [ [[a b] v] [[x₁ y₁] [x₂ y₂] ... [xₙ yₙ]] ]
```

Creates and reports a new fuzzy set with (continuous) universe equal to the interval [*a b*]. The parameter must be a list of numbers of the form: [[[*a b*] *v*] [[x_1 y_1] [x_2 y_2] ... [x_n y_n]]]. The membership function of the reported fuzzy set is a function that assigns the degree of membership v to every input in the interval [*a,b*], except for the inputs x_i in the list [[x_1 y_1] [x_2 y_2] ... [x_n y_n]]. The function assigns the degree of membership y_i to input x_i. For values outside the universe, the membership function is not defined.

```
let tmp fuzzy:interval-with-points-set [ [[0 10] 0.25] [[0 0.5][2 1]] ]
show fuzzy:evaluation-of tmp 0          ;; 0.5 is shown
show fuzzy:evaluation-of tmp 1          ;; 0.25 is shown
show fuzzy:evaluation-of tmp 2          ;; 1 is shown
show fuzzy:evaluation-of tmp 10.5           ;; NaN is shown
```


Fig. 15. Membership function of a fuzzy set created running the code:
fuzzy:plot fuzzy:interval-with-points-set list [[0 100] 0] (n-values 50 [list (2 * ? + 1) 1])

```
fuzzy:interval-with-points-set-with-label lbl [[[a b] v] [[x₁ y₁]... [xₙ yₙ]]]
```

Creates and reports a new interval-with-points fuzzy set with label *lbl*. The parameter *lbl* can be used to retrieve the fuzzy set at a later stage by running the procedure **fuzzy:set-with-label**. See **fuzzy:interval-with-points-set**. The example code below shows the creation of a conventional (crisp) set.

```
let my-new-set fuzzy:interval-with-points-set-with-label
  "odd naturals in [0,100]" list [[0 100] 0] (n-values 50 [list (2 * ? + 1) 1])
set-plot-x-range 0 100
fuzzy:plot my-new-set                           ;; this produces Fig. 15 above
show fuzzy:evaluation-of fuzzy:set-with-label "odd naturals in [0,100]" 19
                                        ;; my-new-set shows 1
show fuzzy:evaluation-of fuzzy:set-with-label "odd naturals in [0,100]" 54
                                        ;; 0 is shown
show fuzzy:evaluation-of fuzzy:set-with-label "odd naturals in [0,100]" 101
                                        ;; NaN is shown
show my-new-set = fuzzy:set-with-label "odd naturals in [0,100]"
                                        ;; true is shown
```

Defuzzification

```
fuzzy:FOM-of set
```

Reports the first value for which the membership function of fuzzy set *set* is maximal. (FOM = First Of Maxima)[4]

```
fuzzy:LOM-of set
```

Reports the last value for which the membership function of fuzzy set *set* is maximal. (LOM = Last Of Maxima)[5]

```
fuzzy:MOM-of set
```

Reports the average of the first and the last values for which the membership function of fuzzy set *set* is maximal, even if the membership function takes lower values in between. (MOM = Middle Of Maxima)

```
fuzzy:MeOM-of set
```

Reports the mean of the values for which the membership function of fuzzy set *set* is maximal.[6] (MeOM = Mean Of Maxima).

```
fuzzy:COG-of set
```

Reports the projection (on the horizontal axis) of the centre of gravity of the area under the membership function of fuzzy set *set*. (COG = Center Of Gravity).

Fig. 16. Membership function of a fuzzy set created running the code:
fuzzy:plot fuzzy:piecewise-linear-set [[0 0][1 1][2 0]]

```
let my-set fuzzy:piecewise-linear-set [[0 0][1 1][2 0]]
fuzzy:plot my-set                    ;; This produces Fig. 16 above.
show fuzzy:FOM-of my-set             ;; 1 is shown
show fuzzy:LOM-of my-set             ;; 1 is shown
show fuzzy:MOM-of my-set             ;; 1 is shown
show fuzzy:MeOM-of my-set            ;; 1 is shown
show fuzzy:COG-of my-set             ;; 1 is shown
```

---

[4] FOM is also known as the Smallest of Maxima.
[5] LOM is also known as the Largest of Maxima.
[6] If there is a continuum of values for which the membership function is maximal, isolated points (which would be intervals of null length) are ignored.

```
let my-set fuzzy:piecewise-linear-set [[0 1][1 0][2 1]]
fuzzy:plot my-set                    ;; This produces Fig. 17 below.
show fuzzy:FOM-of my-set             ;; 0 is shown
show fuzzy:LOM-of my-set             ;; 2 is shown
show fuzzy:MOM-of my-set             ;; 1 is shown
show fuzzy:MeOM-of my-set            ;; 1 is shown
show fuzzy:COG-of my-set             ;; 1 is shown
```
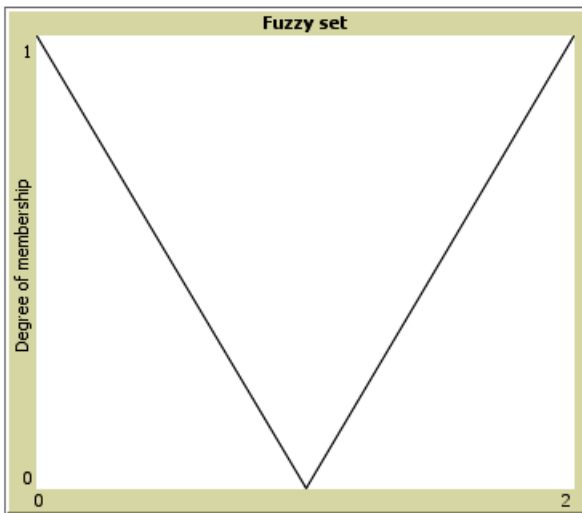

**Fig. 17. Membership function of a fuzzy set created
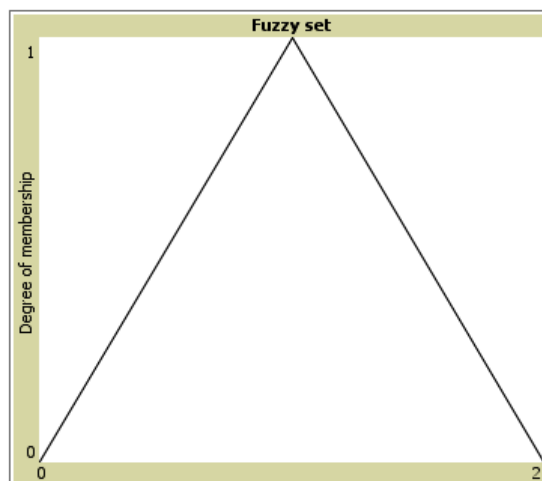running the code:
fuzzy:plot fuzzy:piecewise-linear-set [[0 1][1 0][2 1]]**
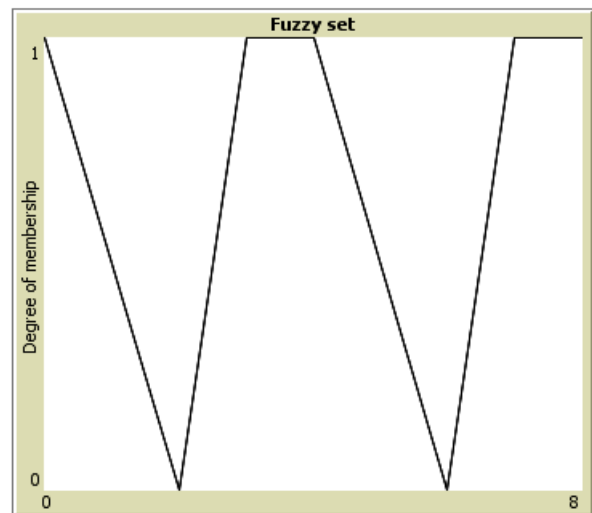
**Fig. 18. Membership function of a fuzzy set created
running the code:
fuzzy:plot fuzzy:piecewise-linear-set
[[0 1][2 0][3 1][4 1][6 0][7 1][8 1]]**

```
let my-set fuzzy:piecewise-linear-set [[0 1][2 0][3 1][4 1][6 0][7 1][8 1]]
fuzzy:plot my-set                    ;; This produces Fig. 18 above.
show fuzzy:FOM-of my-set             ;; 0 is shown
show fuzzy:LOM-of my-set             ;; 8 is shown
show fuzzy:MOM-of my-set             ;; 4 is shown
show fuzzy:MeOM-of my-set            ;; 5.5 is shown
show fuzzy:COG-of my-set             ;; 4.2 is shown
```

Operations with fuzzy sets

fuzzy:min *list-of-sets*   (Another name: **fuzzy:and**)

Creates and reports a new fuzzy set whose membership function is the minimum of the membership functions of all the sets in the input list *list-of-sets*. The universe of the reported set is the intersection of the universes of the sets in the input list *list-of-sets*. For values outside the universe, the membership function is not defined.

```
let low fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
let high fuzzy:piecewise-linear-set [[0 0][3 0][10 1]]
set-plot-x-range 0 10
set-plot-pen-color green      ;; changes the color of the pen we use to plot
fuzzy:plot low                ;; See Fig. 19 below.
set-plot-pen-color red        ;; changes the color of the pen we use to plot
fuzzy:plot high               ;; See Fig. 19 below.
let high-and-low fuzzy:min (list high low)
set-plot-pen-color black      ;; changes the color of the pen we use to plot
fuzzy:plot high-and-low       ;; See Fig. 19 below.
```


Fig. 19. Membership function of two sets (green and red) and its fuzzy:min (black).

```
show fuzzy:evaluation-of high-and-low [3 4 5 6 7]
   ;; [0 0.14285714285714285 0.2857142857142857 0.14285714285714285 0] is shown
show fuzzy:evaluation-of high-and-low -1           ;; NaN is shown
show fuzzy:evaluation-of high-and-low 11           ;; NaN is shown
show fuzzy:MOM-of high-and-low                     ;; 5 is shown
show fuzzy:MeOM-of high-and-low                    ;; 5 is shown
show fuzzy:COG-of high-and-low                     ;; 5 is shown
```

```
fuzzy:max list-of-sets          (Another name: fuzzy:or)
```

Creates and reports a new fuzzy set whose membership function is the maximum of the membership functions of all the sets in the input list *list-of-sets*. The universe of the reported set is the intersection of the universes of the sets in the input list *list-of-sets*. For values outside the universe, the membership function is not defined.

```
let low fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
let high fuzzy:piecewise-linear-set [[0 0][3 0][10 1]]
set-plot-x-range 0 10
set-plot-pen-color green        ;; changes the color of the pen we use to plot
fuzzy:plot low                  ;; See Fig. 20 below.
set-plot-pen-color red          ;; changes the color of the pen we use to plot
fuzzy:plot high                 ;; See Fig. 20 below.
let high-or-low fuzzy:max (list high low)
set-plot-pen-color black        ;; changes the color of the pen we use to plot
fuzzy:plot high-or-low          ;; See Fig. 20 below.
```
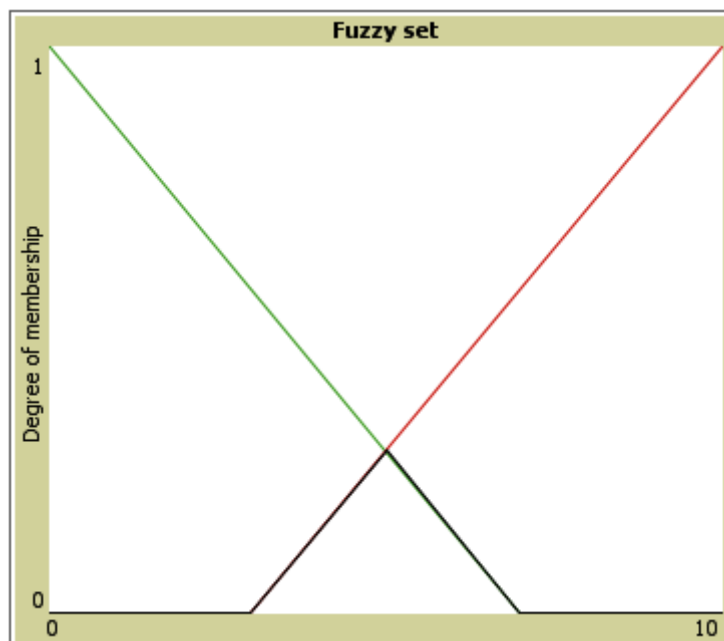

Fig. 20. Membership function of two sets (green and red) and its fuzzy:max (black).

```
show fuzzy:evaluation-of high-or-low [3 4 5 6 7]
                            ;; [0.571 0.429 0.286 0.429 0.571] is shown
show fuzzy:evaluation-of high-or-low -1          ;; NaN is shown
show fuzzy:evaluation-of high-or-low 11          ;; NaN is shown
show fuzzy:MOM-of high-or-low                    ;; 5 is shown
show fuzzy:MeOM-of high-or-low                   ;; 5 is shown
show fuzzy:COG-of high-or-low                    ;; 5 is shown
```

```
fuzzy:sum list-of-sets
```

Creates and reports a new fuzzy set whose membership function is the (clipped) sum of the membership functions of all the sets in the input list *list-of-sets*. If the sum is greater than 1, then the value is clipped to 1. The universe of the reported set is the intersection of the universes of the sets in the input list *list-of-sets*. For values outside the universe, the membership function is not defined.

```
let low fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
let high fuzzy:piecewise-linear-set [[0 0][3 0][10 1]]
set-plot-x-range 0 10
set-plot-pen-color green      ;; changes the color of the pen we use to plot
fuzzy:plot low                ;; See Fig. 21 below.
set-plot-pen-color red        ;; changes the color of the pen we use to plot
fuzzy:plot high               ;; See Fig. 21 below.
let high-plus-low fuzzy:sum (list high low)
set-plot-pen-color black      ;; changes the color of the pen we use to plot
fuzzy:plot high-plus-low      ;; See Fig. 21 below.
```
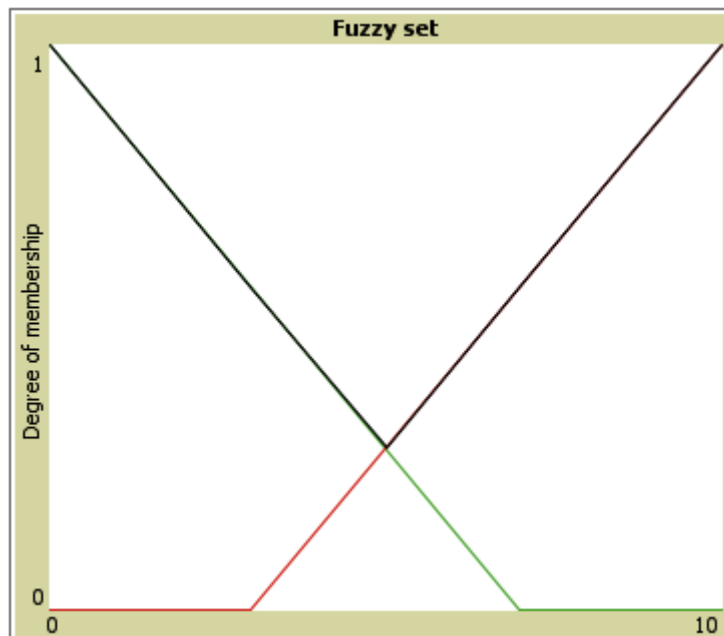

Fig. 21. Membership function of two sets (green and red) and its fuzzy:sum (black).

```
show fuzzy:evaluation-of high-plus-low [2 4 5 6 8]
                          ;; [0.714 0.571 0.571 0.571 0.714] is shown
show fuzzy:evaluation-of high-plus-low -1           ;; NaN is shown
show fuzzy:evaluation-of high-plus-low 11           ;; NaN is shown
show fuzzy:MOM-of high-plus-low                     ;; 5 is shown
show fuzzy:MeOM-of high-plus-low                    ;; 5 is shown
show fuzzy:COG-of high-plus-low                     ;; 5 is shown
show fuzzy:FOM-of high-plus-low                     ;; 0 is shown
show fuzzy:LOM-of high-plus-low                     ;; 10 is shown
```

```
fuzzy:prob-or list-of-sets
```

Creates and reports a new fuzzy set whose membership function is the probabilistic OR of the membership functions of all the sets in the input list *list-of-sets*. For two values *x y*, the probabilistic OR is ($x + y - x*y$). For more than two values the result can be computed recursively, given that the function is associative. The universe of the reported set is the intersection of the universes of the sets in the input list *list-of-sets*. For values outside the universe, the membership function is not defined.

```
let low fuzzy:gaussian-set [2 1 [0 10]]
let high fuzzy:gaussian-set [7 3 [0 10]]
set-plot-x-range 0 10
set-plot-pen-color green       ;; changes the color of the pen we use to plot
fuzzy:plot low                 ;; See Fig. 22 below.
set-plot-pen-color red         ;; changes the color of the pen we use to plot
fuzzy:plot high                ;; See Fig. 22 below.
let high-prob-or-low fuzzy:prob-or (list high low)
set-plot-pen-color black       ;; changes the color of the pen we use to plot
fuzzy:plot high-prob-or-low    ;; See Fig. 22 below.
```
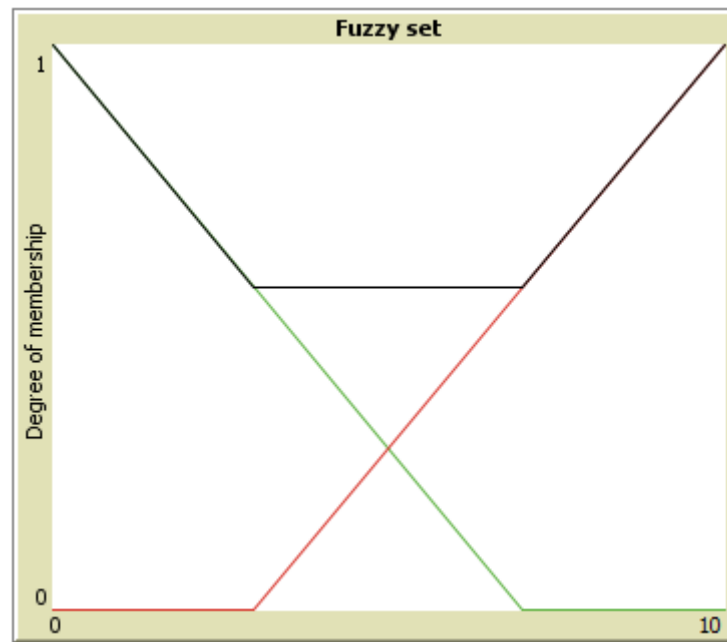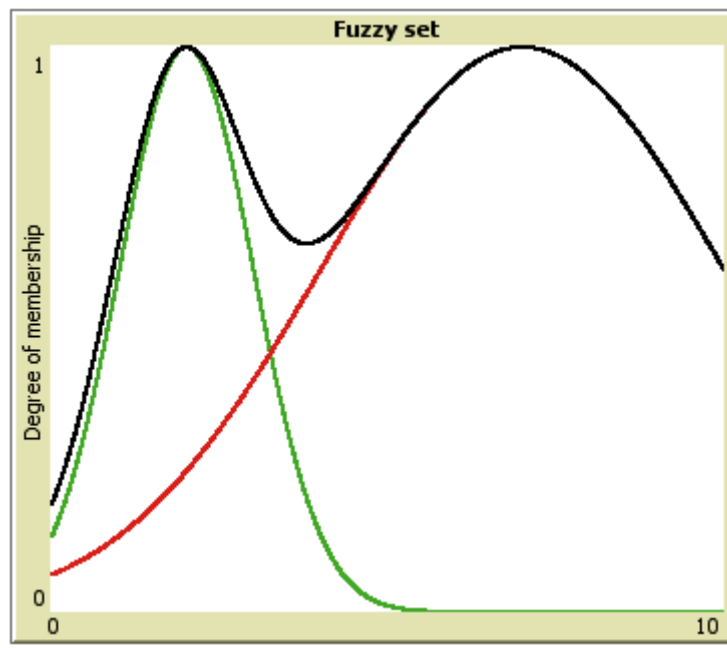

Fig. 22. Membership function of two sets (green and red) and its fuzzy:prob-or (black).

```
show fuzzy:evaluation-of high-prob-or-low [2 4 5 6 7]
                            ;; [1 0.65978 0.80295 0.94598 1] is shown
show fuzzy:evaluation-of high-prob-or-low -1         ;; NaN is shown
show fuzzy:evaluation-of high-prob-or-low 11         ;; NaN is shown
show fuzzy:MOM-of high-prob-or-low                   ;; 4.5 is shown
show fuzzy:MeOM-of high-prob-or-low                  ;; 4.5 is shown
show fuzzy:COG-of high-prob-or-low                   ;; 5.26338 is shown
show fuzzy:FOM-of high-prob-or-low                   ;; 2 is shown
show fuzzy:LOM-of high-prob-or-low                   ;; 7 is shown
```

```
fuzzy:not input-set
```

input-set is a fuzzy set. **fuzzy:not** *input-set* creates and reports a new fuzzy set whose membership function is (1 – MF*input-set*(x)), i.e. the complement of set *input-set*. The universe of the reported set is the universe of *input-set*. For values outside the universe, the membership function is not defined.

```
let low fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
let not-low fuzzy:not low
set-plot-x-range 0 10
set-plot-pen-color green        ;; changes the color of the pen we use to plot
fuzzy:plot low                  ;; See Fig. 23 below.
set-plot-pen-color black        ;; changes the color of the pen we use to plot
fuzzy:plot not-low              ;; See Fig. 23 below.
```
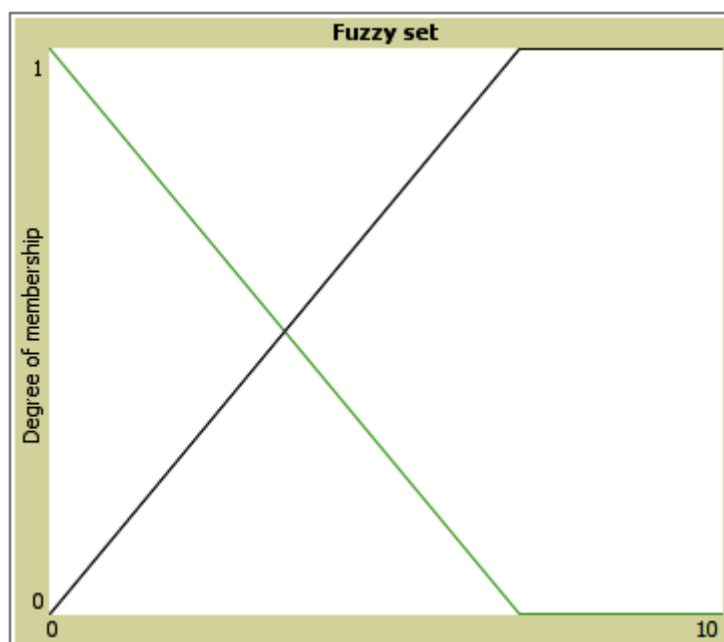

Fig. 23. Membership function of one set (in green) and its fuzzy:not (black).

```
let low-and-not-low fuzzy:and (list low not-low)
show fuzzy:evaluation-of low-and-not-low 3.5  ;; 0.5 is shown
        ;; The contradiction axiom, which dictates that the intersection of a
        ;; set and its complement must be the empty set (Ross, 2010) is not
        ;; necessarily satisfied in fuzzy logic.

let low-or-not-low fuzzy:or (list low not-low)
show fuzzy:evaluation-of low-or-not-low 3.5   ;; 0.5 is shown
        ;; The excluded middle axiom, which dictates that the union of a
        ;; set and its complement must contain all the elements (Ross, 2010)
        ;; is not necessarily satisfied in fuzzy logic.
```

```
fuzzy:truncate input-set c
```

input-set is a fuzzy set and *c* is a number in the interval [0,1]. This procedure creates and reports a new fuzzy set whose membership function is the same as *input-set*'s, but limited by the upper bound *c*. The universe of the reported set is the universe of *input-set*. For values outside the universe, the membership function is not defined.

```
let my-set fuzzy:gaussian-set [7 3 [0 10]]
set-plot-x-range 0 10
set-plot-pen-color red          ;; changes the color of the pen we use to plot
fuzzy:plot my-set               ;; See Fig. 24 below.
let my-set-truncated fuzzy:truncate my-set 0.5
set-plot-pen-color black        ;; changes the color of the pen we use to plot
fuzzy:plot my-set-truncated     ;; See Fig. 24 below.
```
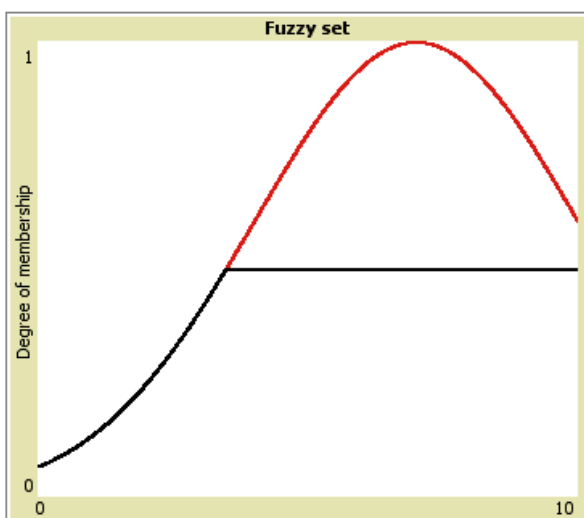

Fig. 24. In black: Membership function of a fuzzy set created running the code:
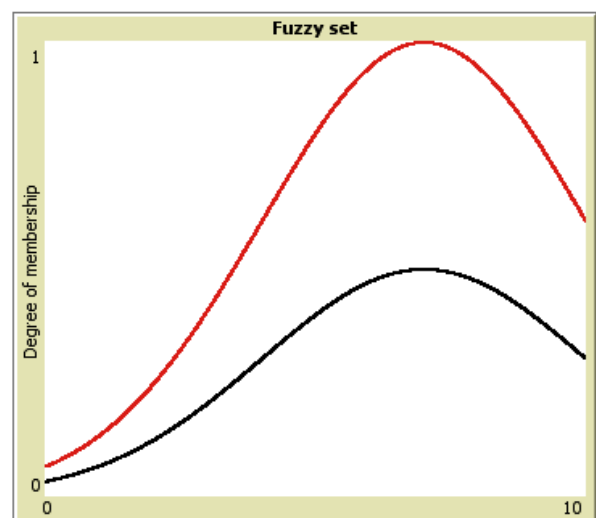**fuzzy:plot fuzzy:truncate fuzzy:gaussian-set [7 3 [0 10]] 0.5**

Fig. 25. In black Membership function of a fuzzy set created running the code:
fuzzy:plot fuzzy:prod fuzzy:gaussian-set [7 3 [0 10]] 0.5

```
fuzzy:prod input-set f
```

input-set is a fuzzy set and *f* is a number in the interval [0,1]. This procedure creates and reports a new fuzzy set whose membership function is *input-set*'s membership function multiplied by factor *f*. Effectively, this operation scales down the original membership function to 100·*f* % of its original size. The universe of the reported set is the universe of *input-set*. For values outside the universe, the membership function is not defined.

```
let my-set fuzzy:gaussian-set [7 3 [0 10]]
set-plot-x-range 0 10
set-plot-pen-color red          ;; changes the color of the pen we use to plot
fuzzy:plot my-set               ;; See Fig. 25 above.
let my-set-prod fuzzy:prod my-set 0.5
set-plot-pen-color black        ;; changes the color of the pen we use to plot
fuzzy:plot my-set-prod          ;; See Fig. 25 above.
```

Hedges

```
fuzzy:power input-set exp
```

input-set is a fuzzy set and *exp* is a positive number. This procedure creates and reports a new fuzzy set whose membership function is the same as *input-set*'s, but raised to the exponent *exp* (i.e. $MF_{input-set}(x)^{exp}$), as long as the value is in the interval [0,1]; otherwise, it is clipped. The universe of the reported set is the universe of *input-set*. For values outside the universe, the membership function is not defined.

```
let hot fuzzy:logistic-set [20 1 .125 [-30 80]]
set-plot-x-range -30 80
set-plot-pen-color green        ;; changes the color of the pen we use to plot
fuzzy:plot hot                  ;; See Fig. 26 below.


let very-hot fuzzy:power hot 2
set-plot-pen-color black        ;; changes the color of the pen we use to plot
fuzzy:plot very-hot             ;; See Fig. 26 below.


let really-hot fuzzy:power hot 6
set-plot-pen-color red          ;; changes the color of the pen we use to plot
fuzzy:plot really-hot           ;; See Fig. 26 below.
```
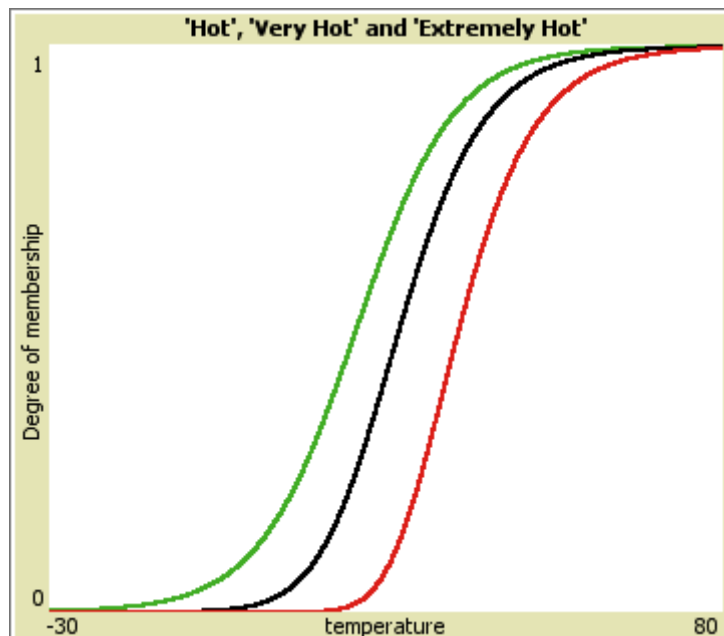

Fig. 26. Membership function of one set (in green), its square (in black), and the green set powered to the 6th (in red).

Rules

```
fuzzy:truncate-rule list-of-2-items consequent-fuzzy-set
(Another name: fuzzy:rule)
```

This procedure can be used to implement simple rules such as "If temperature is hot, set the air-conditioning on high-power", if one wants to use **fuzzy:truncate** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:rule (list temperature hot) high-power
```

The first parameter *list-of-2-items* is a list of 2 items. The first item in the list (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set), to obtain *evaluation*. The procedure creates and reports a new fuzzy set whose membership function is the same as that of *consequent-fuzzy-set* truncated by the calculated number *evaluation*. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is hot, set the air-conditioning on high-power",
;; using fuzzy:truncate to modify the consequent fuzzy set.

let hot fuzzy:piecewise-linear-set [[0 0][27 0][37 1][60 1]]
                                          ;; See Fig. 27 below.
let high-power fuzzy:piecewise-linear-set [[0 0][5 0][10 1]]
                                          ;; See Fig. 27 below.
let temperature 35
    ;; note that temperature could have any value previously computed

let air-conditioning-fuzzy fuzzy:rule (list temperature hot) high-power
    ;; air-conditioning-fuzzy would be the fuzzy set painted in red on Fig. 27.

let air-conditioning-crisp fuzzy:FOM-of air-conditioning-fuzzy
    ;; air-conditioning-crisp would equal 9 in this case
```
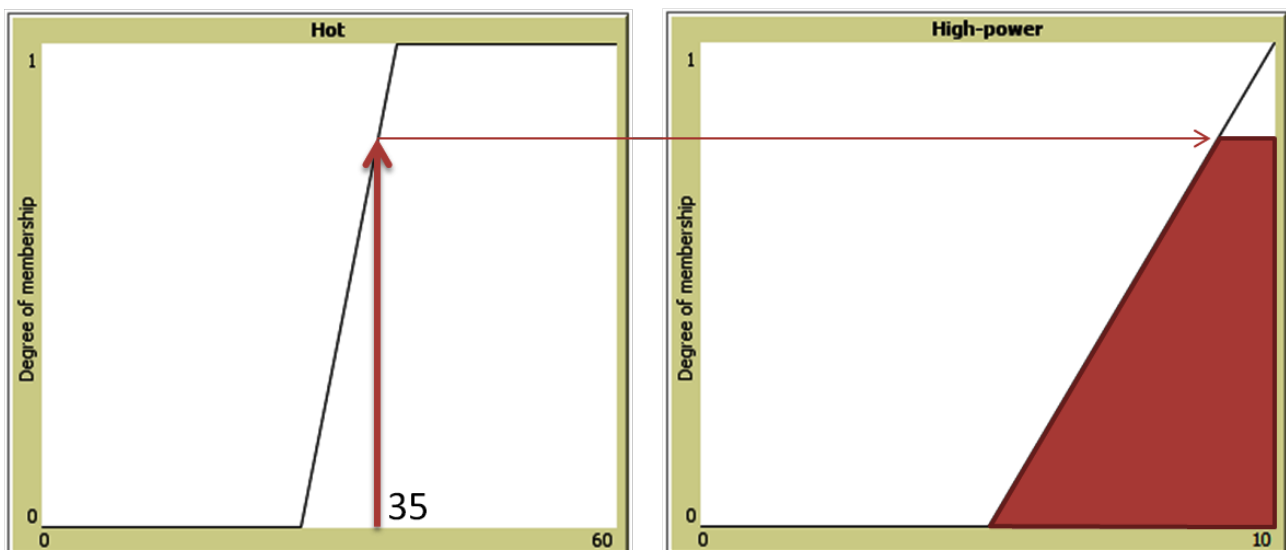

Fig. 27. Illustration of the fuzzy:rule "If temperature is hot, set the air-conditioning on high-power", evaluated at temperature = 35, using fuzzy:truncate to modify the consequent fuzzy set.

```
fuzzy:prod-rule list-of-2-items consequent-fuzzy-set
```

This procedure can be used to implement simple rules such as "If temperature is hot, set the air-conditioning on high-power", if one wants to use **fuzzy:prod** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:prod-rule (list temperature hot) high-power
```

The first parameter *list-of-2-items* is a list of 2 items. The first item in the list (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set), to obtain *evaluation*. The procedure creates and reports a new fuzzy set whose membership function is *A*'s membership function multiplied by factor *evaluation*. In effect, this operation scales down the original membership function to 100·*evaluation* % its original size. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is hot, set the air-conditioning on high-power"
;; using fuzzy:prod to modify the consequent fuzzy set.

let hot fuzzy:piecewise-linear-set [[0 0][27 0][37 1][60 1]]
                                             ;; See Fig. 28 below.


let high-power fuzzy:piecewise-linear-set [[0 0][5 0][10 1]]
                                             ;; See Fig. 28 below.


let temperature 35
    ;; note that temperature could have any value previously computed


let air-conditioning-fuzzy fuzzy:prod-rule (list temperature hot) high-power
    ;; air-conditioning-fuzzy would be the fuzzy set painted in red on Fig. 28.


let air-conditioning-crisp fuzzy:FOM-of air-conditioning-fuzzy
    ;; air-conditioning-crisp would equal 10 in this case
```
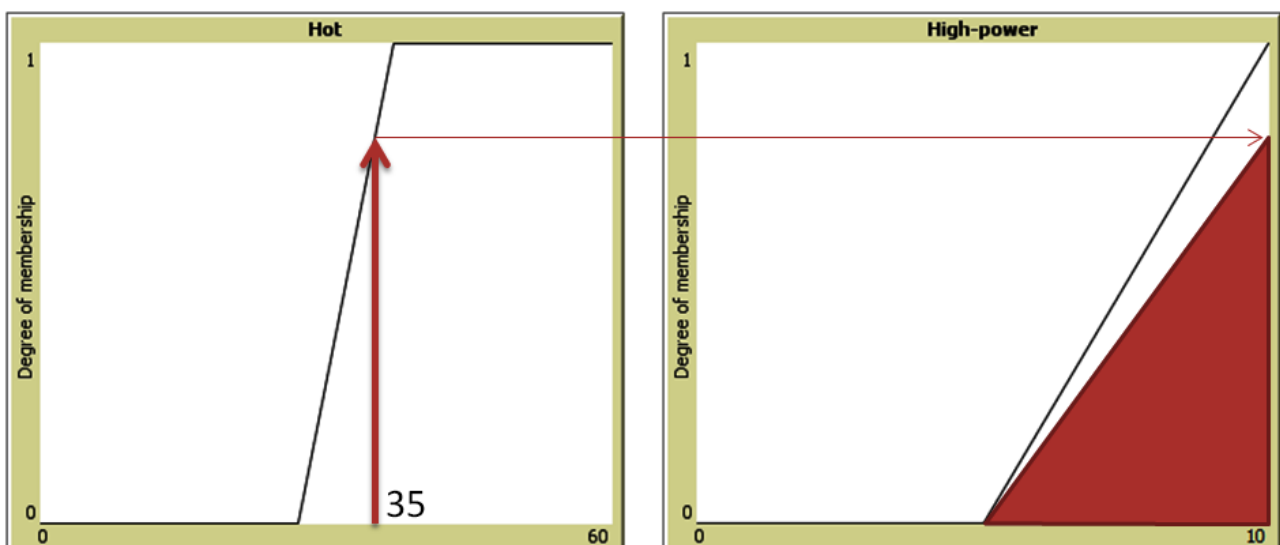

Fig. 28. Illustration of the fuzzy:rule "If temperature is hot, set the air-conditioning on high-power", evaluated at temperature = 35, using fuzzy:prod to modify the consequent fuzzy set.

```
fuzzy:min-truncate-rule list-of-2-item-lists consequent-fuzzy-set
(Another name: fuzzy:and-rule)
```

This procedure can be used to implement rules such as "If temperature is around 20 AND rainfall is low, THEN weather is nice", if one wants to use **fuzzy:min** as AND method and **fuzzy:truncate** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:and-rule (list (list temperature around-20) (list rainfall low)) nice
```

The first parameter *list-of-2-item-lists* is a list of any number of 2-item lists. The first item in each of the 2-item lists (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set) of the 2-item list. This gives a list of *evaluations*. The procedure creates and reports a new fuzzy set whose membership function is the same as that of *consequent-fuzzy-set* truncated by the minimum of all the numbers in the list *evaluations*. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is around 20 and rainfall is low, weather is nice"
;; using fuzzy:min as AND method and fuzzy:truncate to modify the consequent
;; fuzzy set.


let around-20 fuzzy:gaussian-set [20 3 [-10 50]]
                                                ;; See Fig. 29 below.
let low fuzzy:piecewise-linear-set [[0 1][2 0][10 0]]
                                                ;; See Fig. 29 below.
let temperature 22
        ;; note that temperature could have any value previously computed
let rainfall 1.5
        ;; note that rainfall could have any value previously computed


let nice fuzzy:piecewise-linear-set [[0 0][10 1]]
                                                ;; See Fig. 29 below.
let weather-rating-fuzzy fuzzy:and-rule
        (list (list temperature around-20) (list rainfall low) ) nice

    ;; weather-fuzzy would be the fuzzy set painted in red on Fig. 29 below.


let weather-rating-crisp fuzzy:MeOM-of weather-rating-fuzzy
    ;; weather-rating-crisp would equal 6.25 in this case
```
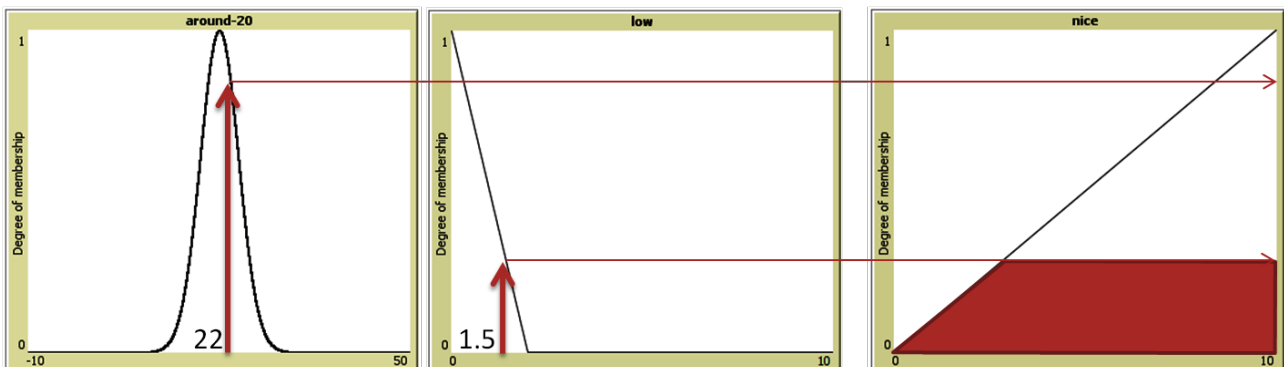

Fig. 29. Illustration of the fuzzy:and-rule "If temperature is around 20 and rainfall is low, weather is nice", evaluated at temperature = 22 and rainfall = 1.5, using fuzzy:min **as AND method and** fuzzy:truncate **to modify the consequent fuzzy set.**

```
fuzzy:min-prod-rule list-of-2-item-lists consequent-fuzzy-set
```

This procedure can be used to implement rules such as "If temperature is around 20 AND rainfall is low, THEN weather is nice", if one wants to use **fuzzy:min** as AND method and **fuzzy:prod** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:min-prod-rule (list (list temperature around-20) (list rainfall low)) nice
```

The first parameter *list-of-2-item-lists* is a list of any number of 2-item lists. The first item in each of the 2-item lists (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set) of the 2-item list. This gives a list of *evaluations*. The procedure creates and reports a new fuzzy set whose membership function is the same as that of *consequent-fuzzy-set* multiplied by the minimum of all the numbers in the list *evaluations*. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is around 20 and rainfall is low, weather is nice"
;; using fuzzy:min as AND method and fuzzy:prod to modify the consequent fuzzy
;; set.

let around-20 fuzzy:gaussian-set [20 3 [-10 50]]
                                              ;; See Fig. 30 below.
let low fuzzy:piecewise-linear-set [[0 1][2 0][10 0]]
                                              ;; See Fig. 30 below.
let temperature 22
      ;; note that temperature could have any value previously computed
let rainfall 1.5
      ;; note that rainfall could have any value previously computed

let nice fuzzy:piecewise-linear-set [[0 0][10 1]]
                                              ;; See Fig. 30 below.
let weather-rating-fuzzy fuzzy:min-prod-rule
      (list (list temperature around-20) (list rainfall low) ) nice

   ;; weather-fuzzy would be the fuzzy set painted in red on Fig. 30 below.

let weather-rating-crisp fuzzy:MeOM-of weather-rating-fuzzy
    ;; weather-rating-crisp would equal 10 in this case
```
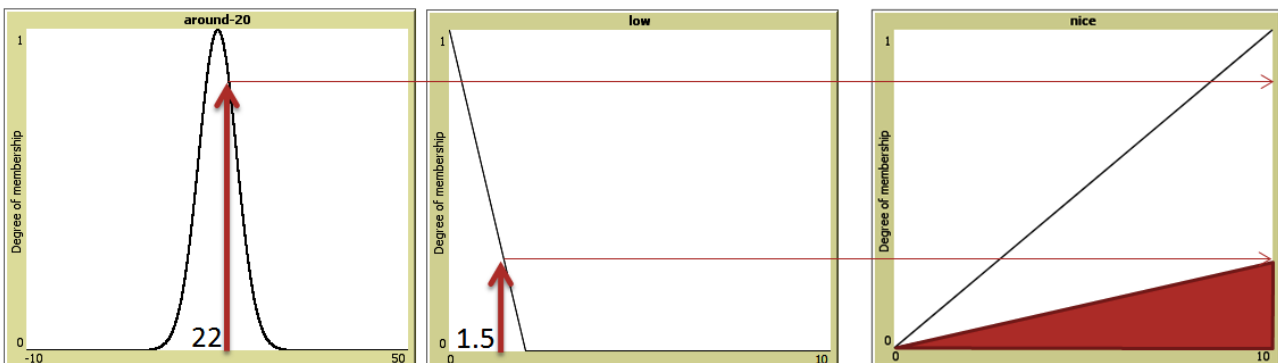

Fig. 30. Illustration of the fuzzy:and-rule "If temperature is around 20 and rainfall is low, weather is nice", evaluated at temperature = 22 and rainfall = 1.5, using `fuzzy:min` as AND method and `fuzzy:prod` to modify the consequent fuzzy set.

```
fuzzy:max-truncate-rule list-of-2-item-lists consequent-fuzzy-set
(Another name: fuzzy:or-rule)
```

This procedure can be used to implement rules such as "If temperature is high OR temperature is low, THEN my comfort is low", if one wants to use **fuzzy:max** as OR method and **fuzzy:truncate** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:or-rule (list (list temperature high) (list temperature low)) low-comfort
```

The first parameter *list-of-2-item-lists* is a list of any number of 2-item lists. The first item in each of the 2-item lists (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set) of the 2-item list. This gives a list of *evaluations*. The procedure creates and reports a new fuzzy set whose membership function is the same as that of *consequent-fuzzy-set* truncated by the maximum of all the numbers in the list *evaluations*. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is high or temperature is low, my comfort is low"
;; using fuzzy:max as OR method and fuzzy:truncate to modify the consequent
;; fuzzy set.

let high fuzzy:piecewise-linear-set [[-10 0][25 0][35 1][50 1]]
                                             ;; See Fig. 31 below.
let low fuzzy:piecewise-linear-set [[-10 1][5 1][15 0][60 0]]
                                             ;; See Fig. 31 below.
let low-comfort fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
                                             ;; See Fig. 31 below.
let temperature 30
      ;; note that temperature could have any value previously computed

let comfort-fuzzy fuzzy:or-rule
      (list (list temperature high) (list temperature low) ) low-comfort

 ;; comfort-fuzzy would be the fuzzy set painted in red on Fig. 31 below.

let comfort-crisp fuzzy:MOM-of comfort-fuzzy
   ;; comfort-crisp would equal 1.75 in this case
```
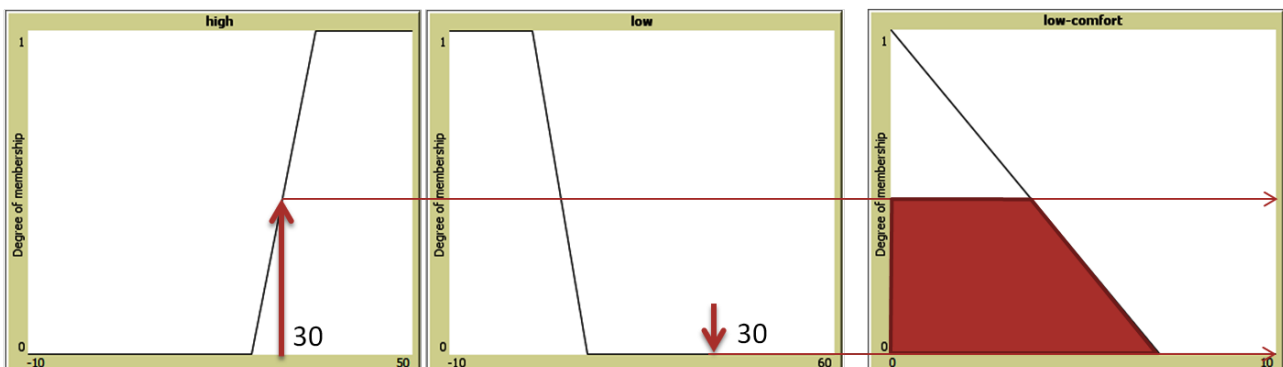

Fig. 31. Illustration of the fuzzy:or-rule "If temperature is high or temperature is low, my comfort is low", evaluated at temperature = 30, using fuzzy:max as OR method and fuzzy:truncate to modify the consequent fuzzy set.

```
fuzzy:max-prod-rule list-of-2-item-lists consequent-fuzzy-set
```

This procedure can be used to implement rules such as "If temperature is high OR temperature is low, THEN my comfort is low", if one wants to use **fuzzy:prod** to modify the consequent fuzzy set. The rule would be written as:

```
fuzzy:max-prod-rule
        (list (list temperature high) (list temperature low)) low-comfort
```

The first parameter *list-of-2-item-lists* is a list of any number of 2-item lists. The first item in each of the 2-item lists (which can be a number or a fuzzy set) will be evaluated in the second item (which is a fuzzy set) of the 2-item list. This gives a list of *evaluations*. The procedure creates and reports a new fuzzy set whose membership function is the same as that of *consequent-fuzzy-set* multiplied by the maximum of all the numbers in the list *evaluations*. The universe of the reported set is the universe of *consequent-fuzzy-set*. For values outside the universe, the membership function is not defined.

```
;; Implementation of the rule:
;; "If temperature is high or temperature is low, my comfort is low"
;; using fuzzy:max as OR method and fuzzy:prod to modify the consequent fuzzy
set.

let high fuzzy:piecewise-linear-set [[-10 0][25 0][35 1][50 1]]
                                          ;; See Fig. 32 below.
let low fuzzy:piecewise-linear-set [[-10 1][5 1][15 0][60 0]]
                                          ;; See Fig. 32 below.
let low-comfort fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
                                          ;; See Fig. 32 below.
let temperature 30
        ;; note that temperature could have any value previously computed

let comfort-fuzzy fuzzy:max-prod-rule
        (list (list temperature high) (list temperature low) ) low-comfort

 ;; comfort-fuzzy would be the fuzzy set painted in red on Fig. 32 below.

let comfort-crisp fuzzy:MOM-of comfort-fuzzy
    ;; comfort-crisp would equal 0 in this case
```
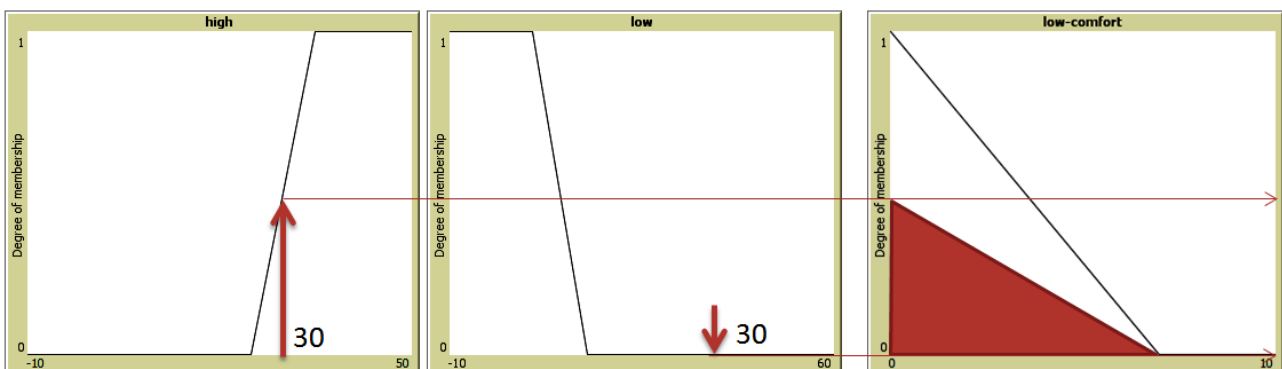

Fig. 32. Illustration of the fuzzy:or-rule "If temperature is high or temperature is low, my comfort is low", evaluated at temperature = 30, using `fuzzy:max` as OR method and `fuzzy:prod` to modify the consequent fuzzy set.

Rule evaluation and aggregation

Rules can be easily combined using so-called aggregation operators. Common choices are **fuzzy:max** (= **fuzzy:or**), **fuzzy:sum**, or **fuzzy:prob-or**. As an example, we include the implementation of the **fuzzy:or** aggregation of the following two rules:

"If temperature is high OR temperature is low, THEN my comfort is low"
"If temperature is average, THEN my comfort is high".

We create a procedure that takes temperature as an input and produces a crisp value for comfort in the interval [0, 10].

```
to-report compute-comfort [temperature]
        ;; Implementation of the fuzzy:or aggregation of the rules:
        ;; "If temperature is high or temperature is low, my comfort is low"
        ;; "If temperature is average, my comfort is high"

 ;; FUZZY SETS
  let high fuzzy:piecewise-linear-set [[-10 0][25 0][35 1][60 1]]
  let low fuzzy:piecewise-linear-set [[-10 1][5 1][15 0][60 0]]
  let avg fuzzy:piecewise-linear-set [[-10 0][10 0][20 1][30 0][60 0]]
  let low-comfort fuzzy:piecewise-linear-set [[0 1][7 0][10 0]]
  let high-comfort fuzzy:piecewise-linear-set [[0 0][3 0][10 1]]

 ;; RULES
  let R1 fuzzy:or-rule
        (list (list temperature high) (list temperature low) ) low-comfort
  let R2 fuzzy:rule
        (list temperature avg) high-comfort

 ;; AGGREGATION
  let comfort-fuzzy fuzzy:or (list R1 R2)

 ;; DEFUZZIFICATION
  let comfort-crisp fuzzy:MOM-of comfort-fuzzy

  report comfort-crisp
end
```

To see a plot of temperature against comfort, you can use the following code

```
let temperatures n-values 70 [? - 10]
let comforts map [compute-comfort ?] temperatures
(foreach temperatures comforts [plotxy ?1 ?2])
```

We advise you to try out different methods of defuzzification and see how different the results are.

Advanced material (optional)

Resolution

The resolution is an integer that measures the precision in any process that involves discretization (e.g. plotting a membership function). The value of the resolution is the number of parts in which an interval of length 1 is divided. The default value is 32. The user can change the value of this variable to e.g. 8 by typing: `fuzzy:set-resolution` 8. It is highly recommended to use numbers that are powers of 2 (i.e. 1, 2, 4, 6, 8, 16, 32, …) to avoid floating-point errors.

Implementation of the extension

Fuzzy sets are implemented as java objects with the following attributes:

```
String description
String label
List membership-function-parameters
double[] universe
Boolean continuous
```

The following describes each of the attributes in turn.

- **description** is a string that specifies the type of fuzzy set (i.e. its type of membership function). It can be "discrete", "continuous", "interval with points", "piecewise-linear", "logistic", "gaussian", "exponential", "power", "truncate", "min", "max", "sum", "prob-or", "not", "empty fuzzy set" or "creator".

- **label** is a string that can be used to retrieve the fuzzy set (once the label of the fuzzy set has been set). This can be done using the reporter **fuzzy:set-with-label** (see section "Retrieval of fuzzy sets").

- **membership-function-parameters** is a list with all the parameters of the membership function. The structure of the list varies depending on the type of fuzzy set, and is explained in detail in the description of the different methods that can be used to create fuzzy sets from scratch.

  As an example, the structure of the **membership-function-parameters** list for piecewise-linear fuzzy sets (such as the one shown in Fig. 1) is a list of points of the form: $[[x_1\ y_1]\ [x_2\ y_2]\ ...\ [x_n\ y_n]]$. The membership function is a piecewise linear function that joins all such points $[x_i\ y_i]$.

- **universe** is a list that specifies the elements of the universe X, i.e. the domain of the membership function. For values outside the universe X, the membership function is not defined. For fuzzy sets with continuous domain, the **universe** is a 2-number list denoting a real interval.1 For fuzzy sets with discrete domain, the **universe** is a non-empty list of numbers (of any length).

- **continuous** is a Boolean variable that equals true if the universe X of the fuzzy set is continuous and false otherwise. As an example, **continuous** is true in fuzzy sets created with the procedure **fuzzy:piecewise-linear-set** and false in fuzzy sets created with the procedure **fuzzy:discrete-numeric-set**.

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